# Polar angle distributions on generalized circles

## Representations and generalizations of the von Mises distribution

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# Content

# From physical problems to circular distributions.

## Some of the basic references
## or... bringing Atomium to Brussels.

Langevin(1905)
von Mises(1918)
Gumbel, Greenwood and Durand(1953)
Fisher(1959)
Mardia(1972)
Batschelet(1981)
Fisher, Lewis and Embleton(1987)
Mardia and Jupp(2000)
Jammalamadaka and SenGupta(2001)
Pewsey, Neuhäuser and Ruxton(2013)

# The conditional offset approach

Mardia(1972)
by a construction following arguments in Fisher(1959)
Jammalamadaka and SenGupta(2001)
Mardia and Jupp(2000)
Shimizu and Iida(2002)
Jones and Pewsey(2005)
Gatto and Jammalamadaka(2007)

# ... the conditional offset approach

Conditional distribution of the directional component of a random vector given its fixed distance from the origin.

Distance from the origin
= radius of the circle where the vector belongs to
= value of the vector's radius variable.

# Generalized circles / radius variables, 1

Power exponential distribution:

$$f_{(X,Y)}(x,y) = C_p \exp\{-\frac{|x|^p}{p} - \frac{|y|^p}{p}\} = C_p \exp\{-\frac{||(x,y||_p^p}{p}\}$$

$p = 1$ ... Laplace distribution, $p = 2$ ... Gaussian distribution

$$||(x,y)||_p = (|x|^p + |y|^p)^{1/p} \text{ p-norm if } p \geq 1$$

$$\frac{(X,Y)}{||(X,Y)||_p} : \Omega \to \{(x,y) \in \mathbb{R}^2 : |x|^p + |y|^p = 1\}$$

**p-circle**

**Generalized radius variable** of $(X, Y)$: $R = ||(X,Y)||_p$.

# Generalized circles, 2

Norm contoured distribution:

$$f_{(X,Y)}(x,y) = C(g, ||.||)g(||(x,y)||),\ (x,y) \in R^2$$

$$||.|| : R^2 \to [0,\infty) \text{ arbitrary norm}$$

$$\frac{(X,Y)}{||(X,Y)||_p} : \Omega \to \{(x,y) \in R^2 : ||(x,y)|| = 1\}$$

**norm-circle**

**Generalized radius variable** of $(X, Y)$: $\quad R = ||(X, Y)||.$

# Generalized radius variables, 3

$K \subset \mathbb{R}^2$ star body having the origin as an inner point

$$h_K((x,y)) = \inf\{\lambda > 0 : (x,y) \in \lambda K\}$$

$r{\cdot}K = \{(x,y) \in \mathbb{R}^2 : h_K((x,y)) \leq r\} =: K(r)$ star ball of star radius r

Topological boundary of $K(r)$

$$= \{(x,y) \in \mathbb{R}^2 : h_K((x,y)) = r\} = S(r)$$

**star circle of star radius** $r,\ S = S(1)$.

**Generalized radius variable** of $(X,Y)$: $\quad R = h_K((X,Y))$.

# Generalized radius variable, 4

**Example** $K$ is a convex body, symmetric w.r.t. the origin
iff
$h_K$ is a norm.

**Example** $K$ elliptical disc ...
... circumscribed by $E_{(a,b)}$... axes-aligned ellipse:

$$E_{(a,b)} = \{(x,y) \in R^2 : (\frac{x}{a})^2 + (\frac{y}{b})^2 = 1\},\ 0 < b \leq a$$

heteroscedastic Gaussian or elliptically contoured distribution

$$R = h_K((X,Y)) = ((\frac{X}{a})^2 + (\frac{Y}{b})^2)^{1/2}$$

# Adapted directional vector variable, 1

$E_{(a,b)}$-generalized trigonometric functions:

$$\cos_{(a,b)}(\phi) = \frac{\cos\phi}{aN_{(a,b)}(\phi)}, \sin_{(a,b)}(\phi) = \frac{\sin\phi}{bN_{(a,b)}(\phi)},$$

$N_{(a,b)}(\phi) = ((\frac{\cos\phi}{a})^2 + (\frac{\sin\phi}{b})^2)^{1/2}$.

$$E_{(a,b)} = \{\begin{pmatrix} a\cos_{(a,b)}(\phi) \\ b\sin_{(a,b)}(\phi) \end{pmatrix}, 0 \leq \phi < 2\pi\}$$

# Adapted directional vector variable, 2

$E_{(a,b)}$-generalized elliptical coordinates $R, \Phi$:

$$X = Ra\cos_{(a,b)}(\Phi),\ Y = Rb\sin_{(a,b)}(\Phi)$$

Inverse transformation:

$$R = h_{K_{(a,b)}}((X, Y)) = ((X/a)^2 + (Y/b)^2)^{1/2},$$ elliptical radius coordinate

$$K_{(a,b)} = \{(x, y) \in R^2 : (\frac{x}{a})^2 + (\frac{y}{b})^2 \leq 1\},\ 0 < b \leq a$$

axes-aligned elliptical disc

$$\tan\Phi = \frac{Y}{X},$$ $\Phi =$ **polar angle coordinate**

# Using heteroscedastic, uncorrelated Gaussian vector

$$(X,Y) \sim \Phi_{\lambda\begin{pmatrix} a\cdot\cos_{(a,b)}(\mu) \\ b\cdot\sin_{(a,b)}(\mu)\end{pmatrix},\frac{1}{\delta}\begin{pmatrix} a^2 & 0 \\ 0 & b^2\end{pmatrix}}$$ for some $\delta > 0, \lambda > 0$

$\Phi$... polar angle coordinate of $(X,Y)$.

Elliptical polar angle: $T(\varphi) = \begin{cases} \arccos[\cos_{(a,b)}(\varphi)] & \text{if } 0 \leq \varphi < \pi \\ 2\pi - \arccos[\cos_{(a,b)}(\varphi)] & \pi \leq \varphi < 2\pi \end{cases}$

**Representation 1:** $f_{T(\Phi)|R}(\psi|r) = vMd_{\kappa,T(\mu)}(\psi), \quad \kappa = \lambda\delta r$

# Using a regular Gaussian vector, 1

$$(X, Y) \sim \Phi_{\nu, \frac{1}{\delta}\Sigma}, \ \Sigma = \begin{pmatrix} \sigma_1^2 & \varrho\sigma_1\sigma_2 \\ \varrho\sigma_1\sigma_2 & \sigma_2^2 \end{pmatrix}$$

$$\alpha = \alpha(\Sigma) = [1 - I_{\{0\}}(\rho)]$$

$$\times \left[\frac{\pi}{4} I_{\{\sigma_2\}}(\sigma_1) + (1 - I_{\{\sigma_2\}}(\sigma_1)) \left(\frac{1}{2} \arctan \frac{2\sigma_1\sigma_2\rho}{\sigma_1^2 - \sigma_2^2} + \frac{\pi}{2} I_{(-\infty,0)}(\rho(\sigma_1 - \sigma_2))\right)\right]$$

$$a = a(\Sigma) = \sqrt{\sigma_1^2 \cos^2\alpha + \sigma_2^2 \sin^2\alpha + 2\rho\sigma_1\sigma_2 \sin\alpha\cos\alpha}\,,$$

$$b = b(\Sigma) = \sqrt{\sigma_2^2 \cos^2\alpha + \sigma_1^2 \sin^2\alpha - 2\rho\sigma_1\sigma_2 \sin\alpha\cos\alpha}\,.$$

# Using a regular Gaussian vector, 2

$$X = \frac{R\cos\Phi}{N_{(a,b)}(\Phi-\alpha)}, \quad Y = \frac{R\sin\Phi}{N_{(a,b)}(\Phi-\alpha)}$$

$$\Phi = \tan\frac{Y}{X} \text{ ... polar angle}$$

$$R = h_{K_{a,b}}((X,Y)) = ||(X,Y)||_2 N_{(a,b)}(\Phi-\alpha)$$

**Representation 2:** $f_{T(\Phi-\alpha)|R}(\psi|r) = vMd_{\kappa,T(\mu-\alpha)}, \ \kappa = \lambda\delta r$

where $\mu \in [-\pi,\pi)$, $\lambda > 0$ are uniquely determined by

$$\nu = \frac{\lambda}{N_{(a,b)}(\mu-\alpha)} \begin{pmatrix} \cos\mu \\ \sin\mu \end{pmatrix}$$

# Using an elliptically contoured distributed vector

$g : [0,\infty) \to [0,\infty)$ ... density generating function $0 < \int_0^\infty rg(r)dr < \infty$ $g$-generalized von Mises density (Jones and Pewsey (2005)):

$$vMd_{g,\lambda,r,\theta}(\psi) = Cg((\lambda^2 + r^2 - 2\lambda r\cos(\psi-\theta))^{1/2}),\ \lambda \geq r$$

$$(X,Y) \sim \Phi_{g,\nu,\Theta}$$

$$\phi_{g,\nu,\Theta}(x,y) = D(g)\det\Theta^{-1/2} g(((x-\nu_1, y-\nu_2)\Theta^{-1}(x-\nu_1,y-\nu_2)^T)^{1/2})$$

**Representation 3:** $f_{T(\Phi-\alpha)|R}(\varphi|r) = vMd_{g,\lambda,r,T(\mu-\alpha)}(\psi)$

$\alpha, \mu, \lambda$ as before

# Common angular and $K$-generalized radius variable

Star-shaped vector distribution:

$$(X,Y) \sim \varphi_{g,K,\nu}(x,y) = C(g,K)g(h_K(\begin{pmatrix} x \\ y \end{pmatrix} - \begin{pmatrix} \nu_1 \\ \nu_2 \end{pmatrix}))$$

General representation: $f_{\Phi|R}(\phi|r) = vMd_{g,K,r,\lambda,\mu}(\phi, r)$

$\lambda, \mu$ are uniquely determined by $\nu = \lambda \begin{pmatrix} \cos_K(\mu) \\ \sin_K(\mu) \end{pmatrix}$

$$vMd_{g,K,r,\lambda,\mu}(\varphi) = D(g,K)R_S^2(\varphi)g(h_K(\begin{pmatrix} r\cos_K(\varphi) - \lambda\cos_K(\mu) \\ r\sin_K(\varphi) - \lambda\sin_K(\mu) \end{pmatrix}))$$

$\{R_S(\varphi)\begin{pmatrix} \cos_K(\varphi) \\ \sin_K(\varphi) \end{pmatrix}\} = S$...unit star circle

# Elliptically contoured Gaussian generalized vMd, 1

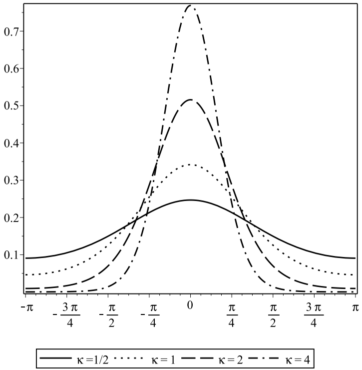


Fig. : unimodal, symmetric

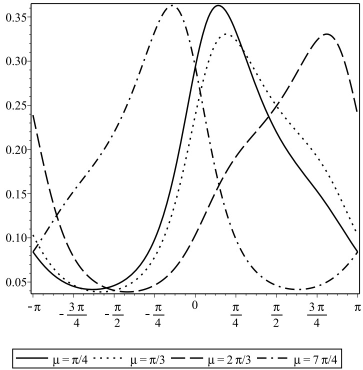


Fig. : unimodal, asymmetric

# Elliptically contoured Gaussian generalized vMd, 2

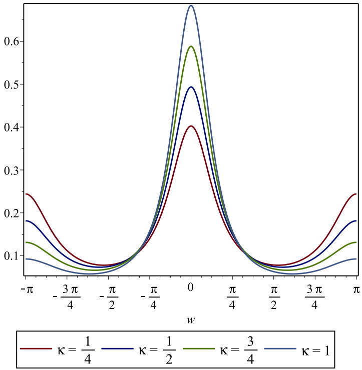


Fig. : bimodal, symmetric

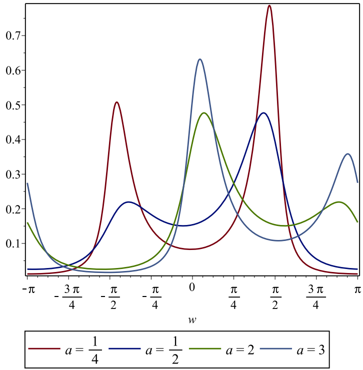


Fig. : bimodal, asymmetric

# Elliptically contoured generalized vMd, Kotz type dgf

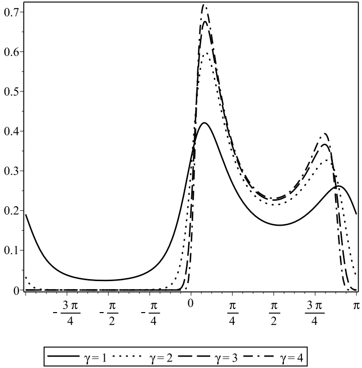


Fig. : bimodal

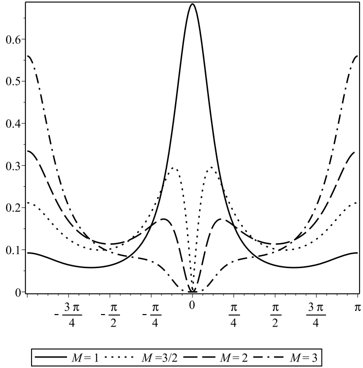


Fig. : multimodal

# Elliptically contoured gen. vMd, Pearson type VII dgf, 1

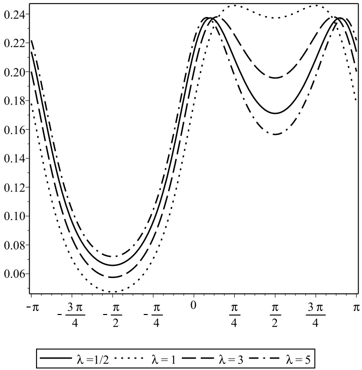


Fig. : bimodal, shift-symmetric

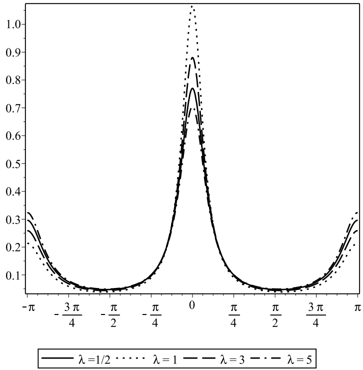


Fig. : bimodal, symmetric

# Elliptically contoured gen. vMd, Pearson type VII dgf, 2

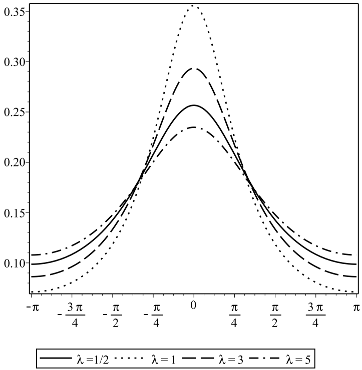


Fig. : unimodal

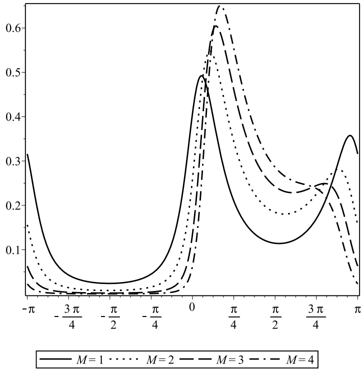


Fig. : bimodal, asymmetric

# Polygonally contoured generalized vMd

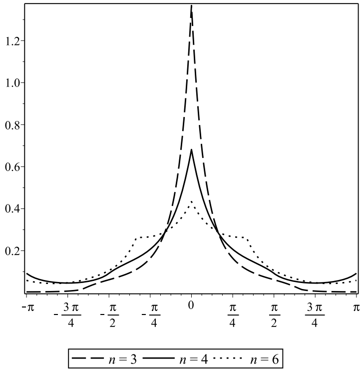


Fig. : $g = g_G$

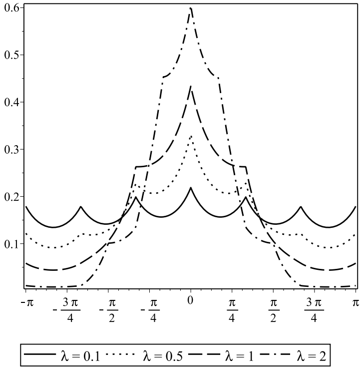


Fig. : $g = g_G, n = 6$

# Non-concentric elliptically contoured generalized vMd

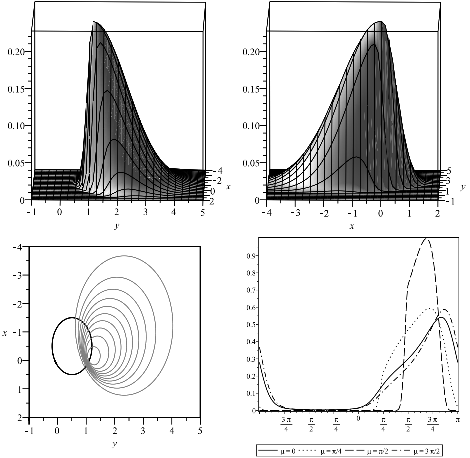


Fig. : $e = 0.5, f = -0.5, \mu = \pi/2, g = g_G$

# The *p*-generalized elliptically contoured vMd, 1

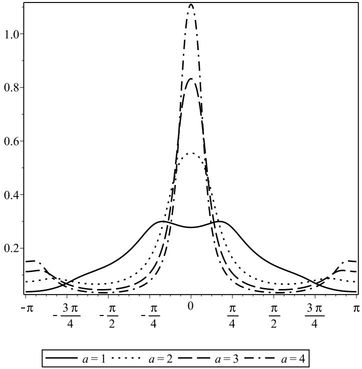


Fig. :
$b = 1, p = 4, \lambda = r = 1, \mu = 0$

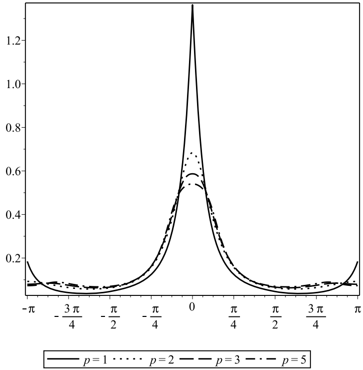


Fig. :
$a = 2, b = 1 = \lambda = r, \mu = 0$

# The *p*-generalized elliptically contoured vMd, 2

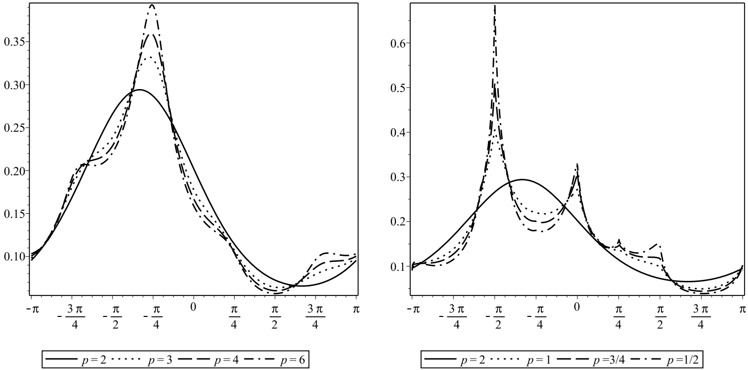


Fig. : $a = b = 1, \lambda = 3/4, r = 1, \mu = 5\pi/3$

Fig. : $a = b = 1, \lambda = 3/4, r = 1, \mu = 5\pi/3$

# Summary

- New representations of the von Mises distribution. The interpretation of the angle, however, is involved.
- New generalizations of the von Mises distribution. Asymmetry and multimodality are possible.
- The conditioning radius variable is constant on norm level sets or star-shaped circles.

- Outlook
  - Multivariate generalizations.
  - Statistics.