Bootstrap Goodness-of-fit Testing for Wehrly–Johnson Bivariate Circular Models

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ADISTA14 (Brussels, Belgium), 21st May 2014

GOBIERNO DE EXTREMADURA
Consejería de Empleo, Empresa e Innovación

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Overview

1. The Wehrly–Johnson class of distributions
2. Tests for toroidal uniformity
3. Bootstrap goodness-of-fit testing
4. Size and power results

The Wehrly–Johnson class of distributions

Definition

A bivariate circular random vector (Θ_1, Θ_2) follows a Wehrly–Johnson (Wehrly & Johnson, 1980) distribution if it has density

$$f(\theta_1, \theta_2) = 2\pi f_1(\theta_1) f_2(\theta_2)\, g(2\pi\{F_2(\theta_2) - qF_1(\theta_1)\}), \quad (1)$$

where f_1 and f_2 are the marginal densities of Θ_1 and Θ_2, F_1 and F_2 their distribution functions, $q \in \{-1, 1\}$, and g is some other circular density which we will refer to as the binding density.

2-fold symmetric cases of density (1)

$$f(\theta_1, \theta_2) = 2\pi f_1(\theta_1) f_2(\theta_2)\, g(2\pi\{F_2(\theta_2) - qF_1(\theta_1)\}) \qquad (1)$$

As in Jones, Pewsey & Kato (2013), we consider cases of (1) that are 2-fold (rotationally) symmetric about (μ_1, μ_2) where μ_p is the mean (and modal) direction of a (reflectively) symmetric unimodal (marginal) distribution with distribution function $F_p : p = 1, 2$. These are obtained if we:

1. define (when g is not circular uniform)

$$F_p(\theta) = \int_{\mu_p}^{\theta} f_p(\phi) d\phi,$$

2. set $\mu_g = 0$.

Plots of BwC and BvM densities

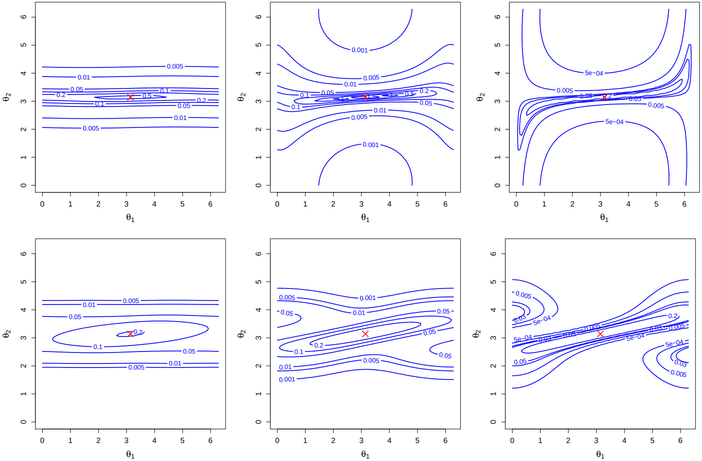

Figure 1 : Contour plots of BwC$(1, \pi, 0.1, \pi, 0.9, 0, \rho_g)$ and BvM$(1, \pi, A^{-1}(0.1), \pi, A^{-1}(0.9), 0, A^{-1}(\rho_g))$ densities with, from left to right, $\rho_g = 0.1, 0.5, 0.9$. The red crosses identify $(\mu_1 = \mu_2 = \pi)$.

Key property

$$f(\theta_1, \theta_2) = 2\pi f_1(\theta_1) f_2(\theta_2)\, g(2\pi\{F_2(\theta_2) - qF_1(\theta_1)\}) \qquad (1)$$

Consider the joint distribution of (Θ_1, Ω), where

$$\Omega = 2\pi\{F_2(\Theta_2) - qF_1(\Theta_1)\}.$$

It is simple to show that

$$f(\theta_1, \omega) = f_1(\theta_1)\, g(\omega),$$

i.e. Θ_1 and Ω are independent.

Consequently, $(2\pi F_1(\Theta_1), 2\pi G(\Omega))$, where G is the distribution function associated with g, is uniformly distributed on the torus.

Wellner tests for toroidal uniformity

Wellner (1979) proposed toroidal equivalents of the Rayleigh and Bingham tests for uniformity:

Rayleigh-type test $T_R = 2n^{-1}|R|^2$,
where $|R|^2 = \sum_{p=1}^{2} a_p^2 + b_p^2$, with $a_p = \sum_{j=1}^{n} \cos\theta_{pj}$ and $b_p = \sum_{j=1}^{n} \sin\theta_{pj}$, $p = 1, 2$.

Bingham-type test $T_B = 4n^{-1}\{(aa)^2 + (ab)^2 + (ba)^2 + (bb)^2\}$,
where $aa = \sum_{j=1}^{n} \cos\theta_{1j} \cos\theta_{2j}$,
$ab = \sum_{j=1}^{n} \cos\theta_{1j} \sin\theta_{2j}$, $ba = \sum_{j=1}^{n} \sin\theta_{1j} \cos\theta_{2j}$
and $bb = \sum_{j=1}^{n} \sin\theta_{1j} \sin\theta_{2j}$.

Wellner and Jupp tests for toroidal uniformity

Wellner's Rayleigh-type and Bingham-type tests are not consistent against all alternatives (as they are Sobolev tests with just one non-zero constant in each of their definitions).

As a remedy to this problem, Jupp (2009) proposed a so-called data-driven Sobolev test that is consistent against all alternatives; with the non-zero constants determined from the data.

Jupp's data-driven test for toroidal uniformity

Jupp's test $T_{\tilde{m}}$, where

$$T_m = \frac{1}{n}\sum_{k=1}^{n}\sum_{j=1}^{n}\left\{\prod_{p=1}^{2} h_m(\theta_{pj},\theta_{pk})\right\} - n,$$

$$h_m(\phi,\psi) = \begin{cases} \frac{\sin((m+1/2)(\phi-\psi))}{\sin((\phi-\psi)/2)}, & \text{if } \phi \neq \psi, \\ 2m+1, & \text{if } \phi = \psi, \end{cases}$$

and $\tilde{m}$ is chosen such that

$$\tilde{m} = \inf\left\{m^* \in \mathbb{N} : PS(m^*) = \sup_{1\leq m\leq L(n)} PS(m)\right\},$$

where

$$PS(m) = T_m - ((2m+1)^2 - 1)\log(n)$$

is a penalized score statistic and L is some suitable function of n.

Testing for toroidal uniformity and goodness-of-fit

Under toroidal uniformity, the sampling distributions of T_R and T_B are both asymptotically χ^2_4, while that of $T_{\tilde{m}}$ is asymptotically χ^2_8.

When testing goodness-of-fit, rather than applying the tests to $(2\pi F_1(\theta_1), 2\pi G(\omega))$-values calculated for known parameter values, we must estimate the parameters of the chosen Wehrly–Johnson model. When applied to values of $(2\pi \hat{F}_1(\theta_1), 2\pi \hat{G}(\hat{\omega}))$, the sampling distributions of T_R, T_B and $T_{\tilde{m}}$ are no longer as specified above and even for relatively large sample sizes can differ substantially from their asymptotic χ^2 distributions under toroidal uniformity.

The obvious computer-intensive strategy to adopt is one incorporating parametric bootstrap simulation.

General bootstrap goodness-of-fit approach

1. Compute MLEs, values of $\hat{\omega}_j = 2\pi\{\hat{F}_2(\theta_{2,j}) - q\hat{F}_1(\theta_{1,j})\}$ and $(2\pi\hat{F}_1(\theta_{1,j}), 2\pi\hat{G}(\hat{\omega}_j))$, $j = 1, ..., n$, and the test statistic value for a test for toroidal uniformity, $\mathcal{T}_0$.
2. Simulate B bootstrap samples from the distribution fitted to the original data in the previous step.
3. For the bth ($b = 1, ..., B$) bootstrap sample, compute MLEs, values of $\tilde{\omega}_j$ and $(2\pi\tilde{F}_1(\theta_{1,j}), 2\pi\tilde{G}(\tilde{\omega}_j))$, $j = 1, ..., n$, and test statistic value of test for toroidal uniformity, $\mathcal{T}_b$.
4. The p-value of the test is the proportion of the $(B + 1)$ $\mathcal{T}$-values that are at least as extreme as $\mathcal{T}_0$.

Transformation to toroidal uniformity for BwC case

As Kato & Pewsey (2013) show, if $(\Theta_1, \Theta_2) \sim$ BwC$(q, \mu_1, \rho_1, \mu_2, \rho_2, 0, \rho_g)$ then

$$(\mathrm{Arg}(C_1), \mathrm{Arg}(C_2)) \ (\mathrm{mod}\ 2\pi),$$

is uniformly distributed on the torus, where

$$C_1 = \frac{(\eta_1 \rho_1 - Z_1)}{(\rho_1 Z_1 - \eta_1)}, \quad C_2 = \frac{(\alpha\beta - Z_2)}{(\overline{\beta} Z_2 - \alpha)},$$

$$\alpha = \frac{\eta_2(\rho_2 \rho_g C_1^{-q} + 1)}{(\rho_2 \rho_g C_1^{q} + 1)}, \quad \beta = \frac{(\rho_2 + \rho_g C_1^{q})}{(\rho_2 \rho_g C_1^{-q} + 1)},$$

$$Z_p = e^{i\Theta_p}, \quad p = 1, 2,$$

$$\eta_p \in \{z \in \mathbb{C}\, ; \, |z| = 1\}, \ \text{with } \mathrm{Arg}(\eta_p)\ (\mathrm{mod}\ 2\pi) = \mu_p, \quad p = 1, 2,$$

and $\overline{z}$ denotes the complex conjugate of z.

Size of goodness-of-fit tests for BwC model

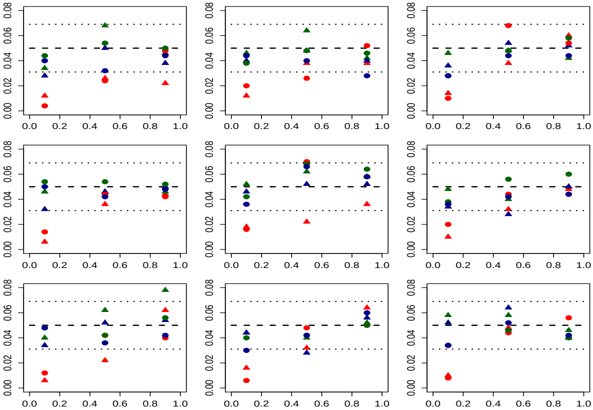

Figure 2 : Estimated size of the Rayleigh-type, Bingham-type and Jupp based goodness-of-fit tests as a function of ρ_g and a nominal significance level of 5%. Rows (columns) are: first, $\rho_1(\rho_2) = 0.1$; second, $\rho_1(\rho_2) = 0.5$; third, $\rho_1(\rho_2) = 0.9$. Each size value was estimated using 500 samples of size 20 (▲) or 50 (●) simulated from the BwC$(1, \pi, \rho_1, \pi, \rho_2, 0, \rho_g)$ distribution and $B = 199$ parametric bootstrap samples simulated from the ML fitted BwC model.

Power against BvM model for assumed BwC

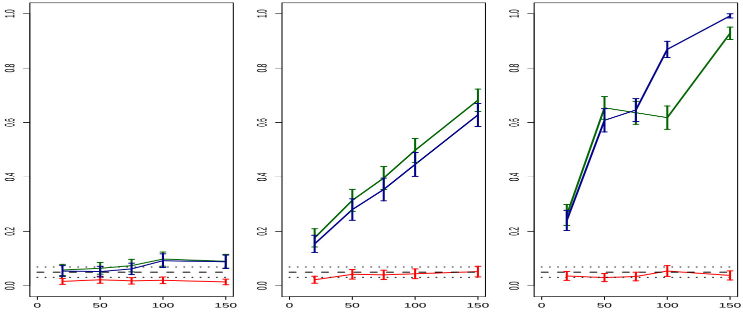

Figure 3 : Estimated power of the Rayleigh-type, Bingham-type and Jupp based goodness-of-fit tests as a function of sample size, n, and a nominal significance level of 5%. Left, $\rho_g = 0.1$; centre, $\rho_g = 0.5$; right, $\rho_g = 0.9$. Each power value was estimated using 500 samples of size n simulated from the BvM$(1, \pi, A^{-1}(0.1), \pi, A^{-1}(0.9), 0, A^{-1}(\rho_g))$ distribution and $B = 199$ parametric bootstrap samples simulated from the ML fitted BwC model.

BwC data, assumed underlying BwC model

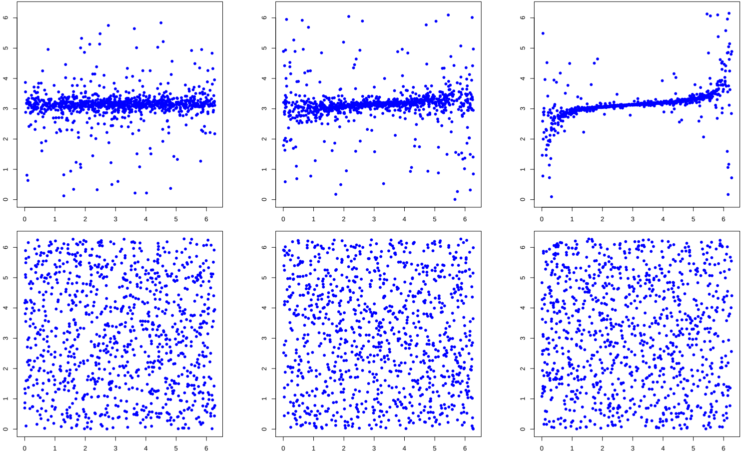

Figure 4 : Top row: 1000 data points simulated from the BwC$(1, \pi, 0.1, \pi, 0.9, 0, \rho_g)$ distribution with $\rho_g = 0.1$ (left), $\rho_g = 0.5$ (centre), $\rho_g = 0.9$ (right). Bottom row: Corresponding (Arg(C_1), Arg(C_2))-values after fitting a BwC model using maximum likelihood.

BvM data, assumed underlying BwC model

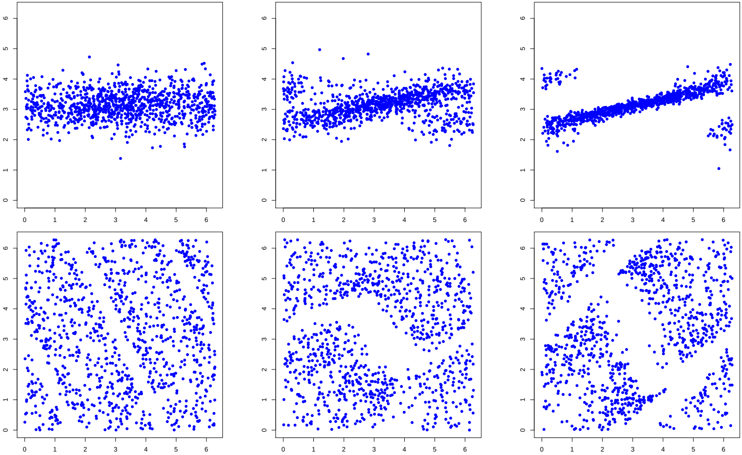

Figure 5 : Top row: 1000 data points simulated from the BvM$(1, \pi, A^{-1}(0.1), \pi, A^{-1}(0.9), 0, A^{-1}(\rho_g))$ distribution with $\rho_g = 0.1$ (left), $\rho_g = 0.5$ (centre), $\rho_g = 0.9$ (right). Bottom row: Corresponding (Arg(C_1), Arg(C_2))-values after fitting a BwC model using maximum likelihood.

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