Universal asymptotics for high-dimensional sign tests

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Brussels, May 2014

Motivation

We consider high-dimensional directional data, that is, with data that live on

$$\mathcal{S}^{p-1} = \left\{ x \in \mathbb{R}^p : \|x\| = \sqrt{x'x} = 1 \right\}, \qquad p \text{ large}.$$

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Why?

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$$\mathcal{S}^{p-1} = \left\{ x \in \mathbb{R}^p : \|x\| = \sqrt{x'x} = 1 \right\}, \qquad p \text{ large}.$$

Why?

(i) Because many applications involve such data.

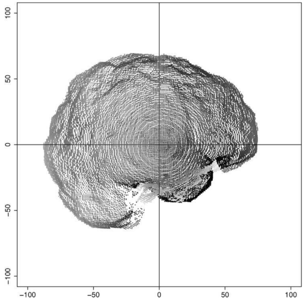
• text analysis: the data may consist of n texts, containing altogether p different words. The observations are then of the form

$$x_i = \begin{pmatrix} x_{i1} \\ \vdots \\ x_{ip} \end{pmatrix}, \quad i = 1, \ldots, n,$$

where x_{ij} is the frequency of the jth word in the ith text. When performing, e.g., clustering, one often replaces x_i with $x_i/\|x_i\|$ so that text length does not play a role; see, among others, Dhillon and Modha (2001), Banerjee et al. (2003).

Motivation

- Dryden (2005) considers an application in brain shape modelling.



Here, observations are of the form

$$x_i = \begin{pmatrix} x_{i1} \\ \vdots \\ x_{ip} \end{pmatrix}, \quad i = 1, \ldots, n,$$

where x_{ij} is the distance between the central landmark and the extremity of brain i in direction j.

- $p = 62{,}501$!
- One is interested in shape, and not in size $\Rightarrow$ one considers $x_i / \|x_i\|$

(ii) Because, in the HD setup, data sometimes automatically become of a directional nature

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Consider $X \sim \mathcal{N}(0, \frac{1}{p} I_p)$.
Then $\|\sqrt{p}\, X\|^2 \sim \chi^2_p$, hence has mean p and variance $2p$.
Therefore, $\|X\|^2$ has mean 1 and variance $2/p$, so that, as $p \to \infty$,

$$\mathrm{E}[(\|X\|^2 - 1)^2] = \mathrm{Var}[\|X\|^2] \to 0.$$

We conclude that X "eventually belongs" to S^{p-1}.

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- This is in line with the fact that $\mathrm{E}[U] = 0$ and $\mathrm{Var}[U] = \frac{1}{p} I_p$ for $U \sim \mathrm{Unif}(\mathcal{S}^{p-1})$

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- This naturally brings sign tests in the picture...

Two low-dimensional tests

Let $X_1, \ldots, X_n$ be i.i.d. with values in $\mathcal{S}^{p-1}$.
We consider the problem of testing for uniformity on $\mathcal{S}^{p-1}$.

The celebrated Rayleigh test rejects the null at asymptotic level α if

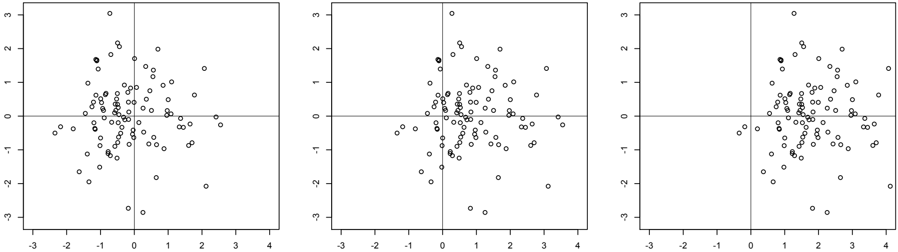
$$T_n = np\|\bar{X}\|^2 = \left(\frac{1}{\sqrt{n}}\sum_{i=1}^n X_i\right)'\left(\frac{1}{p}I_p\right)^{-1}\left(\frac{1}{\sqrt{n}}\sum_{i=1}^n X_i\right) > \chi^2_{p,1-\alpha},$$

where $\chi^2_{d,1-\alpha}$ denotes the $(1-\alpha)$- quantile of the χ^2_d distribution.

In a non-directional framework, this test would be considered as a location test, rejecting the null $\mathcal{H}_0 : \mathrm{E}[X] = 0$ for large values of $\bar{X}$ (this test is valid under sphericity assumptions).

Two low-dimensional tests

→ Increasingly severe alternatives →



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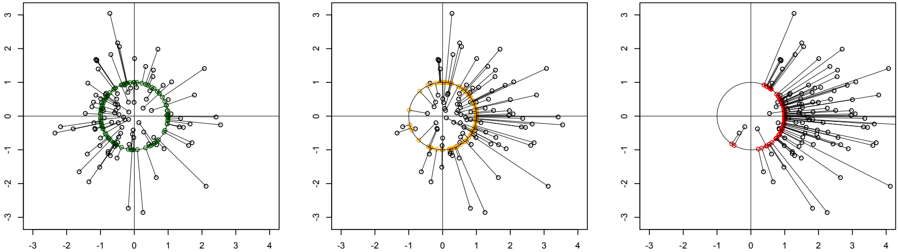

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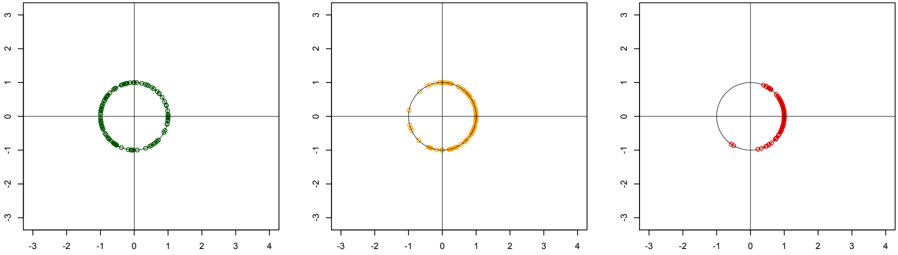

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Two low-dimensional tests

Still for the problem of testing uniformity on $\mathcal{S}^{p-1}$, an alternative test is the Hallin and Paindaveine (2006) test, that rejects the null if

$$\begin{aligned} T_n &= \frac{p(p+2)}{2n} \sum_{i,j=1}^{n} \Big((X_i' X_j)^2 - \frac{1}{p} \Big) > \chi^2_{d(p),1-\alpha} \\ &= \frac{np(p+2)}{2} \Big\| \frac{S}{\mathrm{tr}[S]} - \frac{1}{p} I_p \Big\|^2 \qquad \Big(\text{with } S = \frac{1}{n} \sum_{i=1}^{n} (X_i - 0)(X_i - 0)' \Big), \end{aligned}$$

with $\|A\|^2 = \mathrm{tr}[AA']$ and $d(p) = \frac{p(p+1)}{2} - 1$.

Two low-dimensional tests

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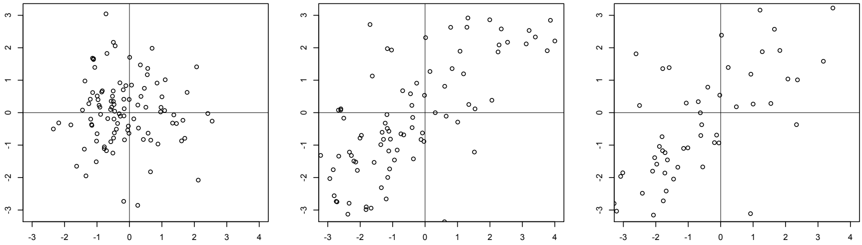

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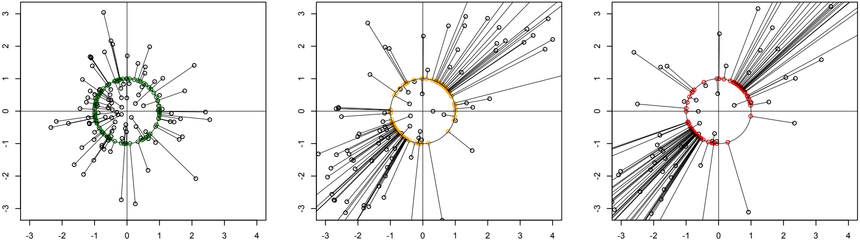

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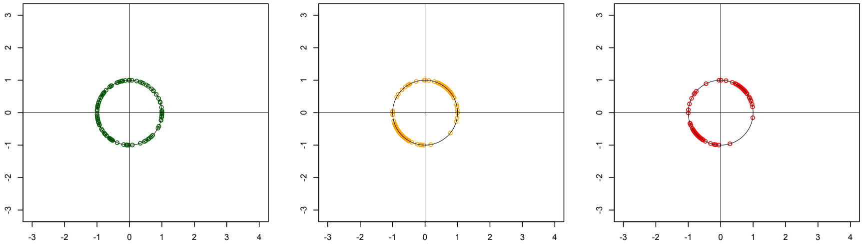

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Two low-dimensional tests

Both tests, and many more from multivariate analysis, rely on a null (fixed-p) asymptotic result of the form

$$T_n \xrightarrow[n\to\infty]{\mathcal{L}} \chi^2_{d(p)},$$

hence lead to rejection (at asymptotic level α) whenever

$$T_n > \chi^2_{d(p),1-\alpha}.$$

Of course, practical implementation of such tests requires $n >> p$.

Such tests therefore are not valid in the HD setup.

(Yet, their fixed-p optimality motivates studying their HD properties...)

The HD case

What would you expect in the HD setup where both $n, p \to \infty$?

Some heuristics...

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What would you expect in the HD setup where both $n, p \to \infty$?

Some heuristics... Assume that $d(p) \to \infty$ as $p \to \infty$.

- Ok for Rayleigh, $d(p) = p$
- Ok for Hallin and Paindaveine (2006), $d(p) = \frac{p(p+1)}{2} - 1$
- Ok for...

What would you expect in the HD setup where both $n, p \to \infty$?

Some heuristics... Assume that $d(p) \to \infty$ as $p \to \infty$.

Since

$$\frac{\chi^2_d - d}{\sqrt{2d}} = \frac{\chi^2_d - \mathrm{E}[\chi^2_d]}{\sqrt{\mathrm{Var}[\chi^2_d]}} \xrightarrow[d\to\infty]{} \mathcal{N}(0,1),$$

one may then expect that the fixed-p asymptotic result

$$T_n \xrightarrow[n\to\infty]{\mathcal{L}} \chi^2_{d(p)}$$

will lead to a double-asymptotic result of the form

$$T_n^{St} = \frac{T_n - d(p)}{\sqrt{2d(p)}} \xrightarrow[n,p\to\infty]{\mathcal{L}} \mathcal{N}(0,1).$$

Intuitively, this should hold if $p = p_n$ is going to ∞ sufficiently slowly.

The HD case

Natural questions are :

- Is this heuristics valid? That is, is there a $(p = p_n) \to \infty$ such that

$$T_n^{St} = \frac{T_n - d(p)}{\sqrt{2d(p)}} \xrightarrow[n,p\to\infty]{\mathcal{L}} \mathcal{N}(0,1) \text{ ?}$$

- How fast may p_n go to infinity?
- "For (p_n) such that the above convergence holds", which test should be favored for fixed (n, p)?

Test 1: reject at asymptotic level α if $\dfrac{T_n - d(p)}{\sqrt{2d(p)}} > \Phi^{-1}(1-\alpha)$.

Test 2: reject at asymptotic level α if

$$T_n > \chi^2_{d(p),1-\alpha}, \quad \text{or equivalently, if} \quad \frac{T_n - d(p)}{\sqrt{2d(p)}} > \frac{\chi^2_{d(p),1-\alpha} - d(p)}{\sqrt{2d(p)}}.$$

The HD case

To fix ideas, we restrict to Rayleigh's test, with test statistic

$$T_n = np_n \|\bar{X}\|^2 = \frac{p_n}{n} \sum_{i,j=1}^{n} X'_{ni} X_{nj},$$

which can be rewritten as

$$T_n = p_n + \frac{2p_n}{n} \sum_{1 \le i < j \le n}^{n} X'_{ni} X_{nj},$$

so that (recall $d(p) = p$ for Rayleigh's test)

$$T_n^{\mathrm{St}} = \frac{T_n - d(p_n)}{\sqrt{2d(p_n)}} = \frac{T_n - p_n}{\sqrt{2p_n}} = \frac{\sqrt{2p_n}}{n} \sum_{1 \le i < j \le n}^{n} X'_{ni} X_{nj}.$$

This is a U-statistic with an order-2 kernel that depends on $p = p_n$.

The HD case

To study the asymptotic behavior of this U-statistic

$$T_n^{\mathrm{St}} = \frac{\sqrt{2p_n}}{n} \sum_{1 \le i < j \le n}^{n} X'_{ni} X_{nj},$$

the key move is to decompose T_n^{St} into

$$T_n^{\mathrm{St}} = \sum_{\ell=1}^{n} D_{n\ell},$$

where the random variables

$$\begin{aligned} D_{n\ell} &= \mathrm{E}_{n\ell}\left[T_n^{\mathrm{St}}\right] - \mathrm{E}_{n,\ell-1}\left[T_n^{\mathrm{St}}\right] \qquad \ell = 1, \ldots, n \\ &= \frac{\sqrt{2p_n}}{n} \sum_{i=1}^{\ell-1} X'_{ni} X_{n\ell} \end{aligned}$$

form a martingale difference process; here, $\mathrm{E}_{n\ell}[\,\cdot\,]$ denotes expectation with respect to $\sigma(X_1, \ldots, X_\ell)$.

The HD case

We can then rely on a CLT for martingale differences, such as the following.

Theorem (Billingsley (1995), Theorem 35.12)

Let $D_{n\ell}$, $\ell = 1, \ldots, n$, $n = 1, 2, \ldots,$ be a triangular array of random variables such that, for any n, $D_{n1}, D_{n2}, \ldots, D_{nn}$ is a martingale difference sequence with respect to some filtration $\mathcal{F}_{n1}, \mathcal{F}_{n2}, \ldots, \mathcal{F}_{nn}$ (with $\mathcal{F}_{n0} := \{\emptyset, \Omega\}$). Assume that $\mathrm{E}[D_{n\ell}^2] < \infty$ for any n, ℓ ,and that

$$\sum_{\ell=1}^{n} \mathrm{E}\big[D_{n\ell}^2 \,|\, \mathcal{F}_{n,\ell-1}\big] \xrightarrow[n\to\infty]{\mathrm{P}} 1 \tag{1}$$

(where $\xrightarrow{\mathrm{P}}$ denotes convergence in probability), and

$$\sum_{\ell=1}^{n} \mathrm{E}\big[D_{n\ell}^2 \, \mathbb{I}[|D_{n\ell}| > \varepsilon]\big] \xrightarrow[n\to\infty]{} 0. \tag{2}$$

Then $\sum_{\ell=1}^{n} D_{n\ell}$ is asymptotically standard normal.

The HD case

After some work to establish (1)-(2), in the present context, we then obtain

Theorem 1

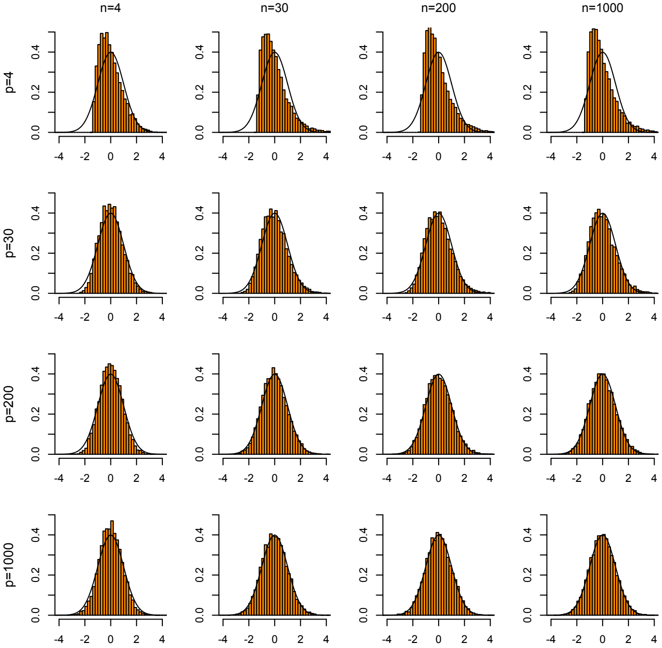
Let p_n be a sequence of positive integers converging to $+\infty$. Assume that X_{ni}, $i = 1, \ldots, n$, $n = 1, 2, \ldots$, is a triangular array such that for any n, the random p_n-vectors X_{ni}, $i = 1, \ldots, n$ are i.i.d. uniform on $\mathcal{S}^{p_n - 1}$. Then

$$T_n^{\mathrm{St}} = \frac{T_n - p_n}{\sqrt{2p_n}} = \frac{\sqrt{2p_n}}{n} \sum_{1 \leq i < j \leq n} X'_{ni} X_{nj} \xrightarrow[n \to \infty]{\mathcal{L}} \mathcal{N}(0, 1).$$

What is interesting is what is *not* there, namely a restriction on how fast p_n should go to infinity with n.

In other words, the result holds as soon as $\min(n, p) \to \infty$ ($\rightsquigarrow$ "universal asymptotics")

This extends to the second test considered
(and actually to various *sign* tests from multivariate analysis).


n=4
n=30
n=200
n=1000
p=4
p=30
p=200
p=1000

The HD case

To the best of our knowledge, this is the first universal (n, p)-asymptotic result. Typically, in previous works,

- one restricts the way p may go to infinity with n. It is standard to have

$$\frac{p_n}{n} \to c \in C$$

  for some convex $C \subset (0, \infty)$ (e.g., $C = (0, 1)$, $[1, \infty)$, etc.)

or

- no such restrictions are imposed, but...
  different (n, p)-regimes lead to different asymptotic distributions;
  see, e.g., Cai and Jiang (2012), Cai, Fan, and Jiang (2013).

Theorem 6 (Extreme Law: Sub-Exponential Case) *Let $p = p_n \to \infty$ satisfy $\frac{\log n}{p} \to 0$ as $n \to \infty$. Then*

(i). $\max_{1 \le i < j \le n} |\Theta_{ij} - \frac{\pi}{2}| \to 0$ *in probability as* $n \to \infty$;

(ii). *As* $n \to \infty$, $2p \log \sin \Theta_{\min} + 4 \log n - \log \log n$ *converges weakly to the extreme value distribution with the distribution function* $F(y) = 1 - e^{-Ke^{y/2}}$, $y \in \mathbb{R}$ *and* $K = 1/(4\sqrt{2\pi})$. *The conclusion still holds if* $\Theta_{\min}$ *is replaced by* $\Theta_{\max}$.

Theorem 8 (Extreme Law: Exponential Case) *Let $p = p_n$ satisfy $\frac{\log n}{p} \to \beta \in (0, \infty)$ as $n \to \infty$, then*

(i). $\Theta_{\min} \to \cos^{-1} \sqrt{1 - e^{-4\beta}}$ *and* $\Theta_{\max} \to \pi - \cos^{-1} \sqrt{1 - e^{-4\beta}}$ *in probability as* $n \to \infty$;

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$$F(y) = 1 - \exp\left\{-K(\beta)e^{(y+8\beta)/2}\right\},\ y \in \mathbb{R},\ \text{where } K(\beta) = \left(\frac{\beta}{8\pi(1 - e^{-4\beta})}\right)^{1/2},$$

and the conclusion still holds if $\Theta_{\min}$ *is replaced by* $\Theta_{\max}$.

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The HD case

Our universal asymptotic results validate the use of two different (asymptotically equivalent) tests, namely

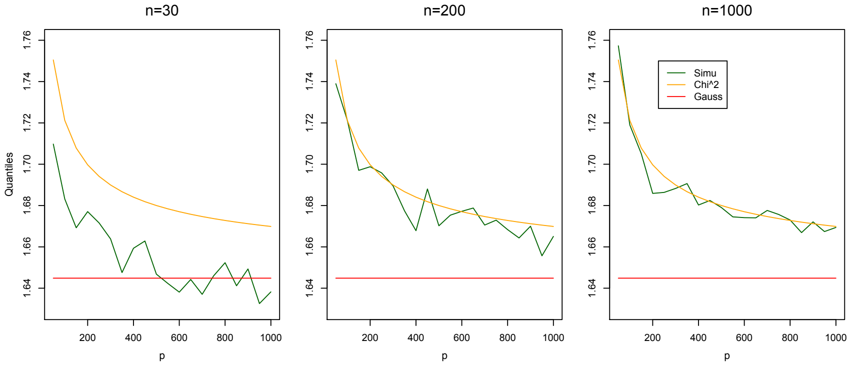
- Test 1: reject at asymptotic level α if

$$T_n^{\mathrm{St}} = \frac{T_n - d(p)}{\sqrt{2d(p)}} > \Phi^{-1}(1-\alpha).$$

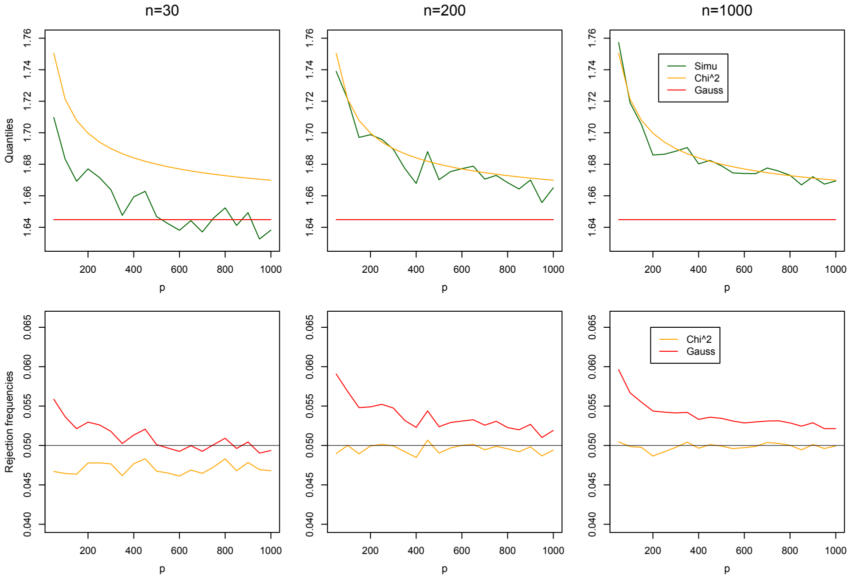
- Test 2: reject at asymptotic level α if $T_n > \chi^2_{d(p),1-\alpha}$, or equivalently, if

$$T_n^{\mathrm{St}} = \frac{T_n - d(p)}{\sqrt{2d(p)}} > \frac{\chi^2_{d(p),1-\alpha} - d(p)}{\sqrt{2d(p)}}.$$

What test should be favored for fixed (n, p)?



From 100,000 replications


n=30
n=200
n=1000
Quantiles
Rejection frequencies
p
Simu
Chi^2
Gauss

Another problem

Ley, Swan, Thiam, and Verdebout (2013) discussed R-estimation in the spherical location problem, that is in the problem of estimating θ from a random sample $X_1, \ldots, X_n$ with common *rotationally symmetric* density

$$x \mapsto c_{p,f}\, f(x'\theta),$$

on $\mathcal{S}^{p-1}$; see Saw (1978).

Paindaveine and Verdebout (2014) recently proposed signed-rank tests for $\mathcal{H}_0 : \theta = \theta_0$, including a sign test that rejects the null whenever

$$T_n = \frac{p-1}{n} \sum_{i,j=1}^{n} U_i'(\theta_0) U_j(\theta_0) > \chi^2_{p-1,1-\alpha},$$

where

$$U_i(\theta_0) = \frac{(I_p - \theta_0\theta_0')X_i}{\|(I_p - \theta_0\theta_0')X_i\|}, \quad i = 1, \ldots, n$$

is the "sign" of the projection of X_i onto the tangent space to $\mathcal{S}^{p-1}$ at θ_0.

Mutatis mutandis, one can establish

Theorem 2

Let p_n be a sequence of positive integers converging to $+\infty$. Assume that X_{ni}, $i = 1, \ldots, n$, $n = 1, 2, \ldots,$ is a triangular array such that for any n, the random p_n-vectors X_{ni}, $i = 1, \ldots, n$ are i.i.d. rotationally symmetric about $\theta_0 \in \mathcal{S}^{p_n-1}$ (with $X_{n1} \neq \theta_0$ a.s.) Then

$$T_n^{\mathrm{St}} = \frac{T_n - (p_n - 1)}{\sqrt{2(p_n - 1)}} \xrightarrow[n\to\infty]{\mathcal{L}} \mathcal{N}(0, 1).$$

Another problem

For the more classical Watson (1983) test, that rejects the null whenever

$$W_n = \frac{n(p-1)\bar{X}'(I_k - \theta_0\theta_0')\bar{X}}{1 - \frac{1}{n}\sum_{i=1}^{n}(X_i'\theta_0)^2} > \chi^2_{p-1,1-\alpha},$$

we can prove the following (where we let $u_{ni} = \sqrt{1 - (X_{ni}'\theta_0)^2}$).

Theorem 3

Let X_{ni}, $i = 1, \ldots, n$, $n = 1, 2, \ldots,$ form a triangular array of random vectors satisfying the following conditions : (i) for any n, $X_{n1}, X_{n2}, \ldots, X_{nn}$ are mutually independent and share a common rotationally symmetric distribution on $\mathcal{S}^{p_n-1}$ with location θ_0; (ii) $p_n \to \infty$ as $n \to \infty$; (iii) $\mathrm{E}[u_{n1}^2] > 0$ for any n; (iv) $\mathrm{E}[u_{n1}^4]/(\mathrm{E}[u_{n1}^2])^2 = o(n)$ as $n \to \infty$. Then

$$W_n^{\mathrm{St}} = \frac{W_n - (p_n - 1)}{\sqrt{2(p_n - 1)}} \xrightarrow[n\to\infty]{\mathcal{L}} \mathcal{N}(0, 1).$$

See Ley, Paindaveine and Verdebout (2014).

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Another problem

No universal consistency

Still...

• Imposing (iii) only excludes the degenerate case for which $X_{n1} = \theta_0$ a.s., which would imply that W_n — hence also W_n^{St} — is not well-defined.

• If (iv) does not hold, we must then have that, for some constant $C > 0$,

$$\mathrm{E}[(X'_{n1}\theta_0)^2] \geq 1 - \frac{C}{\sqrt{n}}$$

for infinitely many n. In the high-dimensional setup considered, this is extremely pathological, since it corresponds to the distribution of X_{n1} concentrating in *one* particular direction — namely, the direction θ_0 — in the expanding Euclidean space $\mathbb{R}^{p_n}$.

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