

Bayes GOF

RA Lockhart

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Bayes assisted goodness-of-fit for von Mises regression

$Richard$ Lockhart $+$ Contreras, Stephens, Sun

Simon Fraser University

ADISTA Brussels, May 22, 2014

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David Blackwell said

I've worked in so many areas — I'm sort of a dilettante. Basically, I'm not interested in doing research and I never have been. I'm interested in understanding, which is quite a different thing.

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- \blacksquare Points of contact with talks of Garcia-Portugués, Jupp, Pewsey, Swan, Verdebout, at least. I am very grateful to have been invited here.
- We do frequentist model assessment via Bayes.
- We construct goodness-of-fit tests for directional regression models.
- **They maximize a certain average power.**
- Some regression models have complete sufficient statistic.
- \blacksquare For these models best test is conditional.
- Implementation via Markov Chain Monte Carlo. $\mathcal{L}_{\mathcal{A}}$
- Methodology allows diagnosis after testing, in principle.

 $\left\{ \begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \end{array} \right\}$

Some general principles -17

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- Admissible nearly implies Bayes.
- Describe model departures via a stochastic process prior on a likelihood ratio.
- To test a goodness-of-fit null, pretend parameters are known, test fit, average results with respect to a posterior on the null.
- To test fit to an assumption about unobservable quantities: pretend they were observed and average results wrt a posterior.

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Von Mises density relative to arc length on the unit circle:

$$
f(y; \tau) = \frac{1}{2\pi I_o(||\tau||)} \exp{\lbrace \tau^{\mathsf{T}} y \rbrace}
$$

 \blacksquare y is a unit vector.

$$
\mathbf{I} \ \tau = \kappa (\cos \theta_0, \sin \theta_0)^T \in \mathbb{R}^2.
$$

modal angle θ_0 .

I

concentration parameter κ .

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Add covariates -15

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Responses yⁱ

Covariates for observation *i* form **matrix** c_i ($p \times 2$). Model: for some $p \times 1$ parameter vector β we have: τ .

$$
\tau_i = c_i^{\prime} \beta.
$$

Likelihood is (ignoring powers of 2π):

$$
L(\beta) = \exp\{\beta' S - \sum \log I_o(\|\tau_i\|)\}
$$

where

$$
S=\sum_i c_i y_i
$$

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is a ($p \times 1$) complete sufficient statistic.

Location Finding **14** and 14

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Guttorp and Lockhart (JASA, 1988) Bayesian aircraft finding.

- Detectors at locations x_1, \ldots, x_n each in \mathbb{R}^2 .
- Lost object at $v \in \mathbb{R}^2$.
- \blacksquare Take bearings y_i at each detector.
- Model y_i as von Mises unit vector with parameter $\mathcal{L}_{\mathcal{A}}$

$$
\tau_i = (v - x_i)\kappa = \left[\begin{array}{cc} 1 & 0 & -x_{i1} \\ 0 & 1 & -x_{i2} \end{array}\right] \left[\begin{array}{c} \kappa v_1 \\ \kappa v_2 \\ \kappa \end{array}\right] \equiv c_i\beta
$$

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Location Finding **-14**

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Guttorp and Lockhart (JASA, 1988) Bayesian aircraft finding.

- Detectors at locations x_1, \ldots, x_n each in \mathbb{R}^2 .
- Lost object at $v \in \mathbb{R}^2$.
- \blacksquare Take bearings y_i at each detector.
- Model y_i as von Mises unit vector with parameter

$$
\tau_i = (v - x_i)\kappa = \left[\begin{array}{cc} 1 & 0 & -x_{i1} \\ 0 & 1 & -x_{i2} \end{array}\right] \left[\begin{array}{c} \kappa v_1 \\ \kappa v_2 \\ \kappa \end{array}\right] \equiv c_i\beta
$$

Narning: unrealistic model chosen to illustrate some ideas. ■ Concentration is higher when object is further from the detector.

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Parameters and statistics

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- **Parameter vector** $\beta = (\nu, \kappa)$ has three components.
- Complete sufficient statistic has three components:

$$
S=(\sum x_i^t y_i,\sum y_i).
$$

- **Peter and I wrote down obvious prior for a different model** and worked with it.
- Next page has posteriors (for x) for old and new models. $\mathcal{L}_{\mathcal{A}}$

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Posteriors for location and the contract of the 11

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Is the von Mises model any good? [I know we don't have enough data to answer the question.]

Ingredients of today's proposal:

- Null hypothesis: the model specified above is right. $\mathcal{L}_{\mathcal{A}}$
- Alternative: von Mises assumption is wrong.
- Define priors on null and alternative hypothesis.
- Maximize average power subject to level α .
- Classical Neyman-Pearson approach.
- \blacksquare Prior makes alternative hypothesis simple.
- Existence of complete sufficient statistic permits solution to optimization problem.

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Priors on the alternative? The state of the set of \sim -9

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- On null $y_i|\beta$ has density $f_i(y_i|\beta)$.
- On alternative y_i has density $g_i(y_i, \beta) = \ell_i(y_i, \beta) f(y_i, \beta)$.
- Describe prior on alternative in two parts:
- First pick parameter value β at random density $\pi_1(\beta)$.

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Then model likelihood ratio ℓ **as stochastic process.**

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 \blacksquare Our choice: for stochastic process Z:

$$
\ell_i(y_i,\beta)=C\exp(aZ(y_i,\beta,x_i)/\sqrt{n})
$$

Many choices for structure of Z: one convenient one is $Z(y, \beta, x) = Z^*(F(y, \beta, x))$

- Take $Z^*(\cdot)$ to be a "stationary" Gaussian process on 'circle'.
- Factor of $n^{-1/2}$ gives contiguous alternatives (Le Cam).
- C is approximated by $\exp(-\int Z^2(y)dy/(2n)).$
- Today's favourite choice: Z has mean 0 and

$$
Cov(Z(s), Z(t)) = \frac{\rho(\cos(2\pi(t-s)) - \rho)}{1 + \rho^2 - 2\rho\cos(2\pi(t-s))}.
$$

Neyman Pearson lemma

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Neyman Pearson: reject for large values of Marginal density on alternative Marginal density on Null

 \blacksquare Has form

 $E[\exp\{T(data, pars)\}]$ data] \times ratio of null marginals

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Neyman Pearson lemma

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- Neyman Pearson: reject for large values of Marginal density on alternative Marginal density on Null
- \blacksquare Has form

 $\mathcal{L}_{\mathcal{A}}$

- E [exp{ T (data, pars)}|data] \times ratio of null marginals
- Need to find least favourable prior on null for denominator. $\mathcal{L}_{\mathcal{A}}$

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Neyman Pearson lemma

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- Neyman Pearson: reject for large values of Marginal density on alternative Marginal density on Null
- \blacksquare Has form

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- E [exp{ T (data, pars)}|data] \times ratio of null marginals
- Need to find least favourable prior on null for denominator. $\mathcal{L}_{\mathcal{A}}$ ■ Can skip this if there is a complete sufficient statistic!

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Conditional tests -6

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 $P_{H_o,\beta}(T\geq t)\leq\alpha\leq P_{H_1}(T\geq t)$

for all β then

If

$$
P_{H_o}(T \geq t | S) \equiv \alpha
$$

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So do conditional test.

Conditional tests -6

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$$
P_{H_o,\beta}(T\geq t)\leq \alpha\leq P_{H_1}(T\geq t)
$$

for all β then

If

$$
P_{H_o}(T \geq t | S) \equiv \alpha
$$

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So do conditional test.

But how do we do it and how well does it work?

Implementation and the state of the state -5

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- How do we do compute p-value?
	- Exact distribution is essentially impossible. $\mathcal{L}_{\mathcal{A}}$
	- Simulation
		- \blacksquare Draw many samples from conditional dist of data given S.

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- **Markov Chain Monte Carlo if that can't be done.**
- **Approximation.**

$MCMC$ and 4

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- Bootstrap
- **[Conclusions](#page-26-0)**
- Often can't easily generate samples $(y_1, \ldots, y_n)|S$. $\mathcal{L}_{\mathcal{A}}$
- So find a Markov Chain whose stationary distribution is

$$
\mathcal{L}(y_1,\ldots,y_n|S)
$$

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and which we can simulate.

■ We have so far used Gibbs sampler.

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Skip this slide. Our method goes like this in iid sampling case:

- Start with original sample $\theta_1, \ldots, \theta_n$ and $S_n = S$.
- **Compute the sufficient statistic** S_3 **for** $\theta_1, \theta_2, \theta_3$ **.**
- **Compute conditional density of** θ_3 **given** S_3 **: joint over** marginal.
- Doesn't depend on von Mises parameter! So do uniform case!
- Joint easy by change of variables. Write S_3 in polar co-ordinates R, Θ.
- **Marginal:** Angle Θ is uniform and independent of R.
- **Marginal: Stephens (1962) uses elliptic integrals to get** density of R.

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- \blacksquare Tried to show conditional tests better than parametric bootstrap.
- Generate many data sets. For each data set:
- Run MCMC to compute conditional p-value.
- Do parametric bootstrap: estimate parameters by ml; compute new test statistic; compare observed test statistic to simulations to get unconditional p-value.
- Plot two p values against each other.

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P -value comparisons -1

[Bayes GOF](#page-0-1) $\frac{0}{1}$ Parametric Bootstrap p-value 0.0 0.0 0.4 0.6 0.0 0.0 0.0 Parametric Bootstrap p−value RA Lockhart **[Outline](#page-0-0)** $0.\overline{8}$ [Conclusions](#page-3-0) $0.\overline{6}$ [von Mises](#page-5-0) **Carpenter Rep** 0.4 **[Conditional](#page-18-0)** 0.2 [Parametric](#page-23-0) Bootstrap 0.0 **[Conclusions](#page-26-0)** 0.0 0.2 0.4 0.6 0.8 1.0 Gibbs Sampling p−value

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- \blacksquare Two methods produce nearly equal P-values.
- So nearly equal levels and powers.
- \blacksquare This is a theorem.
- Starting with work of Lars Holst (Ann Prob).
- **Conditional law of data given S is singular wrt** unconditional law.
- But: conditional dist of gof tests asymptotically same as unconditional.

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Conclusions reiterated, rephrased, augmented, contradicted? The contradicted in the contradicted of the contradicted in the contradicted in the contradicted

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- $\mathcal{L}_{\mathcal{A}}$ Obviously no time to discuss actual data.
- Can maximize average power.
- Choice of "average" (approximate) Gaussian process on $\mathcal{L}_{\mathcal{A}}$ alternative.

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Conclusions reiterated, rephrased, augmented, contradicted? The contradicted in the contradicted of ~ 1

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- Obviously no time to discuss actual data.
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- Choice of "average" (approximate) Gaussian process on $\mathcal{L}^{\mathcal{A}}$ alternative.
- Best unbiased tests are conditional tests. \sim

Conclusions reiterated, rephrased, augmented, contradicted? 1

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- Best unbiased tests are conditional tests. \sim
- Implementation via Markov Chain Monte Carlo.

Conclusions reiterated, rephrased, augmented, contradicted? 1

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- **Obviously no time to discuss actual data.**
- Can maximize average power.
- Choice of "average" (approximate) Gaussian process on $\mathcal{L}^{\mathcal{A}}$ alternative.
- Best unbiased tests are conditional tests. \sim
- Implementation via Markov Chain Monte Carlo.
- The parametric bootstrap nearly implements conditional tests.

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Thanks!

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