Optimal tests for reflective/rotational symmetry

Christophe Ley

joint work with Thomas Verdebout (Université Lille 3)

ADISTA14
May 20–22, 2014, Brussels

Outline

Outline

Distributionology

- Most well-known circular distributions are symmetric, e.g. the von Mises, cardioid, wrapped normal or wrapped Cauchy distributions

Distributionology

- Most well-known circular distributions are symmetric, e.g. the von Mises, cardioid, wrapped normal or wrapped Cauchy distributions
- Mardia (1972) : “symmetrical distributions on the circle are comparatively rare”

Distributionology

- Most well-known circular distributions are symmetric, e.g. the von Mises, cardioid, wrapped normal or wrapped Cauchy distributions
- Mardia (1972) : “symmetrical distributions on the circle are comparatively rare”
- In recent years : growing interest for non-symmetric models, e.g., Umbach and Jammalamadaka (2009, SPL), Kato and Jones (2010, JASA), Abe and Pewsey (2011, Stat. Papers), Jones and Pewsey (2012, Biometrics).
  $\Rightarrow$ Session 7 : Flexible models for circular statistics

Distributionology

- Most well-known circular distributions are symmetric, e.g. the von Mises, cardioid, wrapped normal or wrapped Cauchy distributions
- Mardia (1972) : “symmetrical distributions on the circle are comparatively rare”
- In recent years : growing interest for non-symmetric models, e.g., Umbach and Jammalamadaka (2009, SPL), Kato and Jones (2010, JASA), Abe and Pewsey (2011, Stat. Papers), Jones and Pewsey (2012, Biometrics).
  $\Rightarrow$ Session 7 : Flexible models for circular statistics
- Necessity of tests for circular reflective symmetry (about a known or unknown direction, depending on the experimental design)

Existing tests for reflective symmetry

Essentially 3 proposals (Jupp and Spurr 1983, AoS : test for l-fold symmetry) :

- Schach (1969, Biometrika) : optimal tests for symmetry against rotation alternatives
- Linear tests adapted to the circular setting
- Pewsey (2004, J. Appl. Stat.) : the “b2-star” test, based on the second sine moment about fixed median direction

Existing tests for reflective symmetry

Essentially 3 proposals (Jupp and Spurr 1983, AoS : test for l-fold symmetry) :

- Schach (1969, Biometrika) : optimal tests for symmetry against rotation alternatives
- Linear tests adapted to the circular setting
- Pewsey (2004, J. Appl. Stat.) : the “b2-star” test, based on the second sine moment about fixed median direction

Except for Schach (1969), which is rather a location test, no notion of optimality in these tests.

Existing tests for reflective symmetry

Essentially 3 proposals (Jupp and Spurr 1983, AoS : test for l-fold symmetry) :

- Schach (1969, Biometrika) : optimal tests for symmetry against rotation alternatives
- Linear tests adapted to the circular setting
- Pewsey (2004, J. Appl. Stat.) : the “b2-star” test, based on the second sine moment about fixed median direction

Except for Schach (1969), which is rather a location test, no notion of optimality in these tests.

$\Rightarrow$ Ley and Verdebout (2014, Stat. Sinica) : optimal tests for symmetry about a fixed median direction against k-sine-skewed distributions.

Outline

The family of k-sine-skewed distributions

Let f be a symmetric density on $[-\pi, \pi)$. Inspired by the classical Azzalini-type skew-symmetric distributions on $\mathbb{R}$, Umbach and Jammalamadaka (2009) have turned f into the k-sine-skewed-f density

$$f_{\theta_0,\lambda}(\theta) := f(\theta - \theta_0)(1 + \lambda \sin(k\,(\theta - \theta_0))), \quad \theta \in [-\pi, \pi),$$

with $\theta_0 \in [-\pi, \pi)$ the location parameter and $\lambda \in (-1, 1)$ the skewness parameter. Abe and Pewsey (2011) study the case $k = 1$ and provide conditions when sine-skewed densities are unimodal resp. multimodal.

Test construction step by step

In order to build optimal tests for reflective symmetry $\mathcal{H}_0 : \lambda = 0$ against k-sine-skewed alternatives, we shall proceed in 3 steps :

1. Step 1 : we show that the family of k-sine-skewed distributions for fixed symmetric f is Locally and Asymptotically Normal (LAN)
2. Step 2 : we build, on basis of the LAN property, the efficient parametric test for $\mathcal{H}_0^f : \lambda = 0$ against $\lambda \neq 0$ in

$$f(\theta - \theta_0)(1 + \lambda \sin(k\,(\theta - \theta_0)))$$

3. Step 3 : this parametric test being only valid under f, we render it semi-parametric via a studentization argument

Step 1 : The LAN property for k-sine-skewed densities

Let $\theta_1, \ldots, \theta_n$ be iid $f_{\theta_0,\lambda}$ and denote by $\mathrm{P}^{(n)}_{(\theta_0,\lambda)';f,k}$ their common distribution.

Theorem

The family of k-sine-skewed densities is LAN at $\lambda = 0$ with central sequence

$$\boldsymbol{\Delta}^{(n)}_{f,k}(\theta_0) := n^{-1/2} \sum_{i=1}^{n} \begin{pmatrix} -f'(\theta_i - \theta_0)/f(\theta_i - \theta_0) \\ \sin(k(\theta_i - \theta_0)) \end{pmatrix},$$

and corresponding Fisher information matrix

$$\boldsymbol{\Gamma}_{f,k} := \begin{pmatrix} \Gamma_{f,k;11} & \Gamma_{f,k;12} \\ \Gamma_{f,k;12} & \Gamma_{f,k;22} \end{pmatrix}.$$

More precisely,

$$\log \left(\frac{d\mathrm{P}^{(n)}_{(\theta_0,0)'+n^{-1/2}\boldsymbol{\tau}^{(n)};f,k}}{d\mathrm{P}^{(n)}_{(\theta_0,0)';f,k}} \right) = (\boldsymbol{\tau}^{(n)})'\boldsymbol{\Delta}^{(n)}_{f,k}(\theta_0) - \frac{1}{2}(\boldsymbol{\tau}^{(n)})'\boldsymbol{\Gamma}_{f,k}\boldsymbol{\tau}^{(n)} + o_{\mathrm{P}}(1)$$

and $\boldsymbol{\Delta}^{(n)}_{f,k}(\theta_0) \xrightarrow{\mathcal{L}} \mathcal{N}_2(\mathbf{0}, \boldsymbol{\Gamma}_{f,k})$, *both under* $\mathrm{P}^{(n)}_{(\theta_0,0)';f,k}$ *as* $n \to \infty$.

Singularity of information matrix

The information matrix $\mathbf{\Gamma}_{f,k}$ is singular iff f is a von Mises density.

Singularity of information matrix

The information matrix $\mathbf{\Gamma}_{f,k}$ is singular iff f is a von Mises density.

This is the analog in the circular setting of a (in)famous Fisher information singularity issue described in the linear setting for Azzalini's skew-normal distribution, see *inter alia* Azzalini (1985, Scand. J. Stat.), Pewsey (2000, J. Appl. Stat.), Azzalini and Genton (2008, ISR), Ley and Paindaveine (2010, JMVA), Hallin and Ley (2012, Bernoulli), etc.

Step 2 : Parametric test statistic

Fix $\theta_0 \in [-\pi, \pi)$. From the LAN property, we deduce that the optimal f-parametric test for circular reflective symmetry about θ_0 rejects the null at asymptotic level α whenever the statistic

$$Q_{f;k}^{(n);\theta_0} := \frac{|n^{-1/2}\sum_{i=1}^{n} \sin(k(\theta_i - \theta_0))|}{\Gamma_{f,k;22}^{1/2}}$$

exceeds $z_{\alpha/2}$, the $\alpha/2$ upper quantile of the standard normal distribution.

Step 2 : Parametric test statistic

Fix $\theta_0 \in [-\pi, \pi)$. From the LAN property, we deduce that the optimal f-parametric test for circular reflective symmetry about θ_0 rejects the null at asymptotic level α whenever the statistic

$$Q_{f;k}^{(n);\theta_0} := \frac{|n^{-1/2} \sum_{i=1}^{n} \sin(k(\theta_i - \theta_0))|}{\Gamma_{f,k;22}^{1/2}}$$

exceeds $z_{\alpha/2}$, the $\alpha/2$ upper quantile of the standard normal distribution.

Problem : this statistic, optimal under f, is only valid under that f !

Step 3 : Optimal semi-parametric test statistic via studentization

To overcome this problem, we apply a studentization argument, resulting in a semi-parametric test rejecting the null whenever the studentized statistic

$$Q_k^{*(n);\theta_0} := \frac{|n^{-1/2}\sum_{i=1}^n \sin(k(\theta_i - \theta_0))|}{\left(n^{-1}\sum_{i=1}^n \sin^2(k(\theta_i - \theta_0))\right)^{1/2}}$$

exceeds $z_{\alpha/2}$. This simple test statistic does *no more* depend on f !

Step 3 : Optimal semi-parametric test statistic via studentization

To overcome this problem, we apply a studentization argument, resulting in a semi-parametric test rejecting the null whenever the studentized statistic

$$Q_k^{*(n);\theta_0} := \frac{\left|n^{-1/2} \sum_{i=1}^{n} \sin(k(\theta_i - \theta_0))\right|}{\left(n^{-1} \sum_{i=1}^{n} \sin^2(k(\theta_i - \theta_0))\right)^{1/2}}$$

exceeds $z_{\alpha/2}$. This simple test statistic does *no more* depend on f !

Remarks :

- Nice interpretation of the tests in terms of trigonometric sine moments (link to the work of Batschelet 1965)
- For $k = 2$, we retrieve the “b2-star” test of Pewsey (2004)
- For $k = 1$ and f the uniform density, we retrieve the Rayleigh (1919) test of uniformity against cardioid alternatives.

Properties

Theorem

(i) *under reflective symmetry,* $Q_k^{*(n);\theta_0} \xrightarrow{\mathcal{L}} \mathcal{N}(0,1)$ *as* $n \to \infty$ *;*

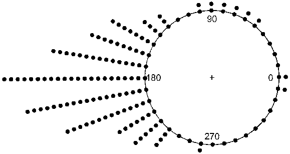
(ii) *under* $\mathrm{P}^{(n)}_{(\theta_0, n^{-1/2}\tau_2^{(n)})'; f, \ell}$ *with* $\ell \in \mathbb{N}_0$, $Q_k^{*(n);\theta_0}$ *is asymptotically normal with mean* $\Gamma^{-1/2}_{f,k;22} C_f(k,\ell)\tau_2$ *and variance* 1*, where* $\tau_2 = \lim_{n\to\infty} \tau_2^{(n)}$ *and* $C_f(k,\ell) := \int_{-\pi}^{\pi} \sin(kt)\sin(\ell t) f(t)\, dt$ *;*

(iii) *for all* f, $Q_k^{*(n);\theta_0} = Q_{f;k}^{(n);\theta_0} + o_{\mathrm{P}}(1)$ *as* $n \to \infty$ *under* $\mathrm{P}^{(n)}_{(\theta_0,0)'; f, k}$*, so that the studentized test is uniformly (in* f *) the best for testing circular reflective symmetry against* k*-sine-skewed alternatives.*

Test	$n = 30/n = 100$	$n = 30/n = 100$	$n = 30/n = 100$	$n = 30/n = 100$
	1-sine-skewed f_{VM_1}			
	$\lambda = 0$	$\lambda = 0.2$	$\lambda = 0.4$	$\lambda = 0.6$
$\phi_1^{*(n);0}$	.047/.048	.110/.266	.311/.779	.608/.988
$\phi_2^{*(n);0}$	.051/.053	.063/.090	.101/.228	.156/.449
$\phi_3^{*(n);0}$	.046/.054	.047/.053	.051/.062	.056/.074
$\phi_{\mathrm{modrun}}^{(n)}$	.054/.054	.064/.072	.112/.156	.204/.375
	2-sine-skewed f_{VM_1}			
	$\lambda = 0$	$\lambda = 0.2$	$\lambda = 0.4$	$\lambda = 0.6$
$\phi_1^{*(n);0}$	.048/.048	.064/.103	.107/.254	.186/.491
$\phi_2^{*(n);0}$	.049/.049	.115/.295	.347/.825	.669/.994
$\phi_3^{*(n);0}$	.047/.047	.060/.097	.102/.242	.173/.482
$\phi_{\mathrm{modrun}}^{(n)}$	.048/.048	.056/.067	.081/.122	.131/.244
	3-sine-skewed f_{VM_1}			
	$\lambda = 0$	$\lambda = 0.2$	$\lambda = 0.4$	$\lambda = 0.6$
$\phi_1^{*(n);0}$	.046/.052	.050/.052	.057/.063	.058/.080
$\phi_2^{*(n);0}$	.051/.050	.060/.098	.104/.240	.176/.482
$\phi_3^{*(n);0}$	.047/.048	.117/.290	.340/.828	.676/.995
$\phi_{\mathrm{modrun}}^{(n)}$	.051/.046	.059/.065	.083/.132	.128/.280

Test	$n = 30/n = 100$	$n = 30/n = 100$	$n = 30/n = 100$	$n = 30/n = 100$
		Moebius transformed f_{VM_1}		
	$\lambda = 0$	$\lambda = 0.2/3$	$\lambda = 0.4/3$	$\lambda = 0.2$
$\phi_1^{*(n);0}$	.051/.051	.074/.127	.142/.351	.239/.639
$\phi_2^{*(n);0}$	.050/.049	.082/.153	.169/.453	.304/.776
$\phi_3^{*(n);0}$	.047/.048	.082/.154	.163/.460	.302/.771
$\phi_{\mathrm{modrun}}^{(n)}$	.049/.053	.057/.059	.074/.074	.086/.116
		Moebius transformed $f_{\mathrm{VM}_{10}}$		
	$\lambda = 0$	$\lambda = 0.02$	$\lambda = 0.04$	$\lambda = 0.06$
$\phi_1^{*(n);0}$	.046/.046	.092/.215	.244/.641	.464/.937
$\phi_2^{*(n);0}$	.047/.048	.092/.215	.245/.644	.466/.938
$\phi_3^{*(n);0}$	.047/.048	.093/.215	.247/.646	.469/.940
$\phi_{\mathrm{modrun}}^{(n)}$	.050/.052	.069/.081	.133/.204	.237/.457

The Jander (1957) experiment



- 730 red wood ants placed individually in the center of an arena with a black target positioned at an angle of 180°
- initial direction upon release recorded to the nearest 10°
- each dot represents the direction chosen by 5 ants
- experimental design suggests a natural fixed median direction.

Pewsey (2004) rejects the null of symmetry. Abe and Pewsey (2011) find that neither symmetric nor 1-sine-skewed densities fit well the data set.

- For k=1, we get a p-value of 0.7781, indicating indeed no preference for 1-sine-skewed laws.

- For k=1, we get a p-value of 0.7781, indicating indeed no preference for 1-sine-skewed laws.
- However, in view of the multi-modality of the data set, the more appropriate tests correspond to $k = 2$ or 3.

- For k=1, we get a p-value of 0.7781, indicating indeed no preference for 1-sine-skewed laws.
- However, in view of the multi-modality of the data set, the more appropriate tests correspond to $k = 2$ or 3.
- The respective p-values are 0.0107 and 0.0131, tending to reject symmetry and showing that the data set should be modeled by a multi-modal skew distribution.

Extension to the center unspecified problem

- This situation is of course much more delicate, as θ_0 needs to be estimated.

Extension to the center unspecified problem

- This situation is of course much more delicate, as θ_0 needs to be estimated.
- The Fisher information matrix $\boldsymbol{\Gamma}_{f,k}$ is rarely diagonal $\Rightarrow$ the cost of not knowing θ_0 entails that a location perturbation has the **same** asymptotic impact on the central sequence for skewness as a skewness perturbation.

Extension to the center unspecified problem

- This situation is of course much more delicate, as θ_0 needs to be estimated.
- The Fisher information matrix $\mathbf{\Gamma}_{f,k}$ is rarely diagonal $\Rightarrow$ the cost of not knowing θ_0 entails that a location perturbation has the **same** asymptotic impact on the central sequence for skewness as a skewness perturbation.
- We can take this impact into account thanks to the asymptotic linearity (under the null)

$$\mathbf{\Delta}_{f,k}^{(n)}(\theta_0 + n^{-1/2}\tau_1^{(n)}) = \mathbf{\Delta}_{f,k}^{(n)}(\theta_0) - \mathbf{\Gamma}_{f,k}(\tau_1^{(n)}, 0)' + o_{\mathrm{P}}(1).$$

- Taking $\tau_1^{(n)} = n^{1/2}(\hat{\theta}_0^{(n)} - \theta_0)$ yields the f-efficient central sequence for skewness

$$n^{-1/2}\sum_{i=1}^{n}\left(\sin(k(\theta_i - \theta_0)) + \frac{\Gamma_{f,k;12}}{\Gamma_{f,k;11}} f\,'(\theta_i - \theta_0)/f(\theta_i - \theta_0)\right),$$

which is no longer correlated with the central sequence for location BUT strongly depends on f now...

Outline

1. Motivation : symmetric or not ?
2. Optimal tests for circular reflective symmetry about a fixed direction
3. Outlook : optimal tests for rotational symmetry

Skew-rotational distributions

Contrarily to the circular case, except for mixtures there is not yet a "rush" for flexible models on spheres $\mathcal{S}^{p-1}$, $p \geq 3$.

To thwart this deficiency, our aim is to flexibilize (resp. skew) rotationally symmetric distributions (Saw 1978, Biometrika)

$$\mathbf{x} \mapsto c_{f,p} f(\mathbf{x}'\boldsymbol{\theta}), \quad \mathbf{x} \in \mathcal{S}^{p-1},$$

where the *angular function* $f : [-1, 1] \to \mathbb{R}^+$ is absolutely continuous and $c_{f,p}$ is a normalizing constant.

Skew-rotational distributions

Contrarily to the circular case, except for mixtures there is not yet a "rush" for flexible models on spheres $\mathcal{S}^{p-1}$, $p \geq 3$.

To thwart this deficiency, our aim is to flexibilize (resp. skew) rotationally symmetric distributions (Saw 1978, Biometrika)

$$\mathbf{x} \mapsto c_{f,p} f(\mathbf{x}'\boldsymbol{\theta}), \quad \mathbf{x} \in \mathcal{S}^{p-1},$$

where the *angular function* $f : [-1, 1] \to \mathbb{R}^+$ is absolutely continuous and $c_{f,p}$ is a normalizing constant.

A rotationally symmetric random vector $\mathbf{X}$ admits the *tangent-normal decomposition*

$$\mathbf{X} = (\mathbf{X}'\boldsymbol{\theta})\boldsymbol{\theta} + (1 - (\mathbf{X}'\boldsymbol{\theta})^2)^{1/2}\mathbf{S}_{\boldsymbol{\theta}}(\mathbf{X})$$

with sign vector $\mathbf{S}_{\boldsymbol{\theta}}(\mathbf{X})$ uniformly distributed on $\mathcal{S}^{p-1}(\boldsymbol{\theta}^{\perp})$.

Skew-rotational distributions

Contrarily to the circular case, except for mixtures there is not yet a "rush" for flexible models on spheres $\mathcal{S}^{p-1}$, $p \geq 3$.

To thwart this deficiency, our aim is to flexibilize (resp. skew) rotationally symmetric distributions (Saw 1978, Biometrika)

$$\mathbf{x} \mapsto c_{f,p} f(\mathbf{x}'\boldsymbol{\theta}), \quad \mathbf{x} \in \mathcal{S}^{p-1},$$

where the *angular function* $f : [-1, 1] \to \mathbb{R}^+$ is absolutely continuous and $c_{f,p}$ is a normalizing constant.

A rotationally symmetric random vector $\mathbf{X}$ admits the *tangent-normal decomposition*

$$\mathbf{X} = (\mathbf{X}'\boldsymbol{\theta})\boldsymbol{\theta} + (1 - (\mathbf{X}'\boldsymbol{\theta})^2)^{1/2}\mathbf{S}_{\boldsymbol{\theta}}(\mathbf{X})$$

with sign vector $\mathbf{S}_{\boldsymbol{\theta}}(\mathbf{X})$ uniformly distributed on $\mathcal{S}^{p-1}(\boldsymbol{\theta}^{\perp})$.

Idea : break rotational symmetry at the level of $\mathbf{S}_{\boldsymbol{\theta}}(\mathbf{X})$.

Definition

Let $\Pi : \mathbb{R} \to [0,1]$ be a skewing function, that is, a monotone increasing continuous function satisfying $\Pi(-y) + \Pi(y) = 1$ for all $y \in \mathbb{R}$. Consider moreover an a.e.-continuous mapping $m : [-1,1] \to \mathbb{R}$, which we call modulating function. Then the spherical distribution with pdf

$$\mathbf{x} \mapsto f_{\boldsymbol{\theta},\boldsymbol{\delta},p}(\mathbf{x}) := 2c_{f,p} f(\mathbf{x}'\boldsymbol{\theta})\Pi(m(\mathbf{x}'\boldsymbol{\theta})\boldsymbol{\delta}'(\mathbf{I} - \boldsymbol{\theta}\boldsymbol{\theta}')\mathbf{x}), \quad \mathbf{x} \in \mathcal{S}^{p-1},$$

is skew-rotational with skewness parameter $\boldsymbol{\delta} \in \mathbb{R}^p$.

This density can be rewritten as ($\Rightarrow$ breaking rotational symmetry)

$$2c_{f,p} f(\mathbf{x}'\boldsymbol{\theta})\Pi(m(\mathbf{x}'\boldsymbol{\theta})(1 - (\mathbf{x}'\boldsymbol{\theta})^2)^{1/2}\boldsymbol{\delta}'\mathbf{S}_{\boldsymbol{\theta}}(\mathbf{x})).$$

The presence of $(1 - (\mathbf{x}'\boldsymbol{\theta})^2)^{1/2}$ suggests a hidden dependence on $\mathbf{x}'\boldsymbol{\theta}$ inside the skewing function ; whence the idea of the modulating function $m(\mathbf{x}'\boldsymbol{\theta})$.

Properties of our model :

- The complicated task of calculating a normalizing constant is not present here !
- Skewness can be modulated along the vector $\boldsymbol{\theta}$.
- If m is odd, then anti-podal symmetry is preserved.
- As $||\boldsymbol{\delta}|| \to \infty$, we obtain half-$f$ distributions.
- Fisher information singularity discussion.
- In the circular setting : k-sine-skewed densities are a particular instance of our construction

Future research

Besides further investigating properties of our models, we want to

(i) investigate by means of information criteria in how far this model improves on the classical ones for diverse data sets

(ii) propose optimal tests for rotational symmetry against skew-rotational alternatives.

for your attention and for having come to Brussels !